# Notes on the application of Lax–Milgram to divergence form equations

February 25, 2016

Recall that we are trying to solve the equation $Lu - \sigma u = F$ weakly in $\Omega$ for $u \in W_0^{1,2}(\Omega)$ and for $F = D_i f^i + g$. Consider the associated bilinear form

$$\mathcal{L}_\sigma(u,v) := \int_\Omega (a^{ij} D_j u D_i v + b^i u D_i v - c^j D_j u v + (\sigma - d)uv)dx.$$

Observe that

$$\mathcal{L}_\sigma : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$$

is a bounded bilinear functional, i.e.

$$|\mathcal{L}_\sigma(u,v)| \leq C_1 \|u\|_{W^{1,2}(\Omega)} \|v\|_{W^{1,2}(\Omega)}$$

(Recall that the $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ norms are the same). Moreover, in lecture, we have seen that $\mathcal{L}_\sigma$ is coercive, at least for $\sigma$ sufficiently large:

$$\mathcal{L}_\sigma(u,u) \geq C_2 \|u\|^2_{W^{1,2}(\Omega)}.$$

We would like to apply Lax–Milgram to $\mathcal{L}_\sigma$. We thus see that for any bounded linear $F : W_0^{1,2}(\Omega) \to \mathbb{R}$, there is a unique $u \in W_0^{1,2}(\Omega)$ so that

$$\mathcal{L}_\sigma(u,v) = F(v)$$

for all $v \in W_0^{1,2}(\Omega)$. In particular, for any such $F$, there is $u \in W_0^{1,2}(\Omega)$ so that

$$\int_\Omega (a^{ij} D_j u D_i v + b^i u D_i v - c^j D_j u v + (\sigma - d)uv)dx = F(v)$$

In particular, we may take any functional of the form

$$F(u) = \int_\Omega (f^i D_i u - gu)dx.$$

for $f^i, g \in L^2(\Omega)$. Thus, we have found $u \in W_0^{1,2}(\Omega)$ solving $L_\sigma u = D_i f^i + g$ weakly for $f^i, g \in L^2(\Omega)$. However, it is more convenient to think of $F(u)$ as an arbitrary element of $W_0^{1,2}(\Omega)^*$. By this, we mean just the space of distributions $w$ that define a bounded linear operator on $W_0^{1,2}(\Omega)$ <u>when using the $L^2$-inner pairing</u>. Notice that $D_i f^i + g$ is precisely such a distribution.

Thus, for $F \in W_0^{1,2}(\Omega)^*$, Lax–Milgram gives us $u \in W_0^{1,2}(\Omega)$ so that

$$\mathcal{L}_\sigma(u, \cdot) = F(\cdot)$$

as elements of $W_0^{1,2}(\Omega)^*$. This map $F \mapsto u$ can be seen to be linear and (by part of the conclusion of Lax–Milgram) bounded. We thus call it $L_\sigma^{-1} : W_0^{1,2}(\Omega)^* \to W_0^{1,2}(\Omega)$. The proof then can go on precisely as in class.

The confusing point is that the dual space (in the usual, abstract sense) to $W_0^{1,2}(\Omega)$ is again itself, because the dual space to a Hilbert space is itself. However, here we're considering the dual with respect to the $L^2$-inner product, so what we get is "different," and we write it as $W_0^{1,2}(\Omega)^*$. It is clear that $L^2(\Omega) \subset W_0^{1,2}(\Omega)^*$ because $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ (what does this mean?).

If you're confused by this it may help, or completely confuse you, to think about the fact that any infinite separable Hilbert space is isometric to $\ell^2(\mathbb{Z})$. So, to say that $\mathcal{H}_1$ is the dual to $\mathcal{H}_2$ makes no sense. We need a way to pair them! A common thing to do is to say that $\mathcal{H}$ is dual to itself with the pairing being the $\mathcal{H}$ inner product. But obviously $\ell^2(\mathbb{Z})$ is also dual to any separable Hilbert space, but there's a totally non-cannonical pairing!